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THE TRILINEAR BINARY FORM AS A CUBIC SURFACE

BY

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Jessie Marie Jacobs

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TABLE OF CONTENTS.

		Page
I.	Historical Introduction	1
II.	Development of the coordinate system connecting the cubic surface and the trilinear form	3
III.	A discussion of the group of the lines	4
IV.	Determination of the equations of fifteen tritangent planes	8
V.	Formation of a new canonical form of the cubic surface	16
VI.	Determination of the remaining thirty tritangent planes	19

I. INTRODUCTION - HISTORICAL

Morgan and Young in a recent memoir* have discussed geometries in which they consider the properties of figures left invariant by certain groups of Cremona transformations. If, in a plane, one has two pencils of lines on two fixed points, U_1 and U_2, every point of the plane in general determines a unique line in each pencil and conversely. Exceptional cases occur either when the given point is U_1 or U_2 and thus a unique line of the pencil is not determined; or when the lines of the pencils pass through U_1 and U_2 and thus a unique point is not determined. By fixing a projectivity in each pencil, a transformation of the points of the plane is thus established. All possible pairs of such projectivities form a group of transformations which transforms into itself the system of all conics on the two fixed points U_1 and U_2. If we give to the lines on U_1, the parameter $x_1 : x_2$ and to those on U_2, the parameter $y_1 : y_2$, then such a system of conics is given by the totality of bilinear binary forms $(\alpha x)(\beta y) = 0$. This is merely a projective version of the inversive geometry, which is of great importance in the theory of functions.

As an extension to an S_3, Morgan and Young consider pencils of planes on three non-concurrent axes U_1, U_2, U_3. Every point of space in general determines a plane in each pencil and conversely. The exceptional cases are noted later. Any set of three projectivities in the three pencils of planes will determine a transformation of the points of space. The group of all such transformations transforms into itself the system of twisted

* Trans. of the Amer. Math. Soc., vol. 17, July 1916, pp. 233-244.

cubics having U_1, U_2, U_3 as bisecants. Such a cubic curve is the extension of the conic in the plane and can be represented as the intersection of a quadric $(\alpha x)(\beta y) = 0$ on U_1 and U_2 with a quadric $(\beta' y)(\gamma z) = 0$ on U_2 and U_3. What seems to be a more natural extension to an S_3 can be developed from the trilinear binary form $(\alpha x)(\beta y)(\gamma z) = 0$, representing the system of cubic surfaces containing three fixed lines.

This problem has been handled synthetically by London*. The purpose of this dissertation is to treat the problem analytically by means of a coordinate system properly chosen with reference to the underlying group. The use of these coordinates is exemplified by the determination of the equations of the forty-five tritangent planes. The subject will be discussed under the following five heads.

1. Development of the coordinate system and the connection between the cubic surface and the trilinear form.

2. A discussion of the group of the lines showing how the solution of a cubic and a quadratic isolates all the lines. The connection between the cubic and quadratic and the group of the lines. A proof of the identity of the discriminants of a certain set of three quadratics.

3. Determination of the equation of fifteen tritangent planes directly from the roots of the cubic and trilinear form.

4. Formation of a new canonical form of the cubic surface and development of a second coordinate system which gives more definite expressions for the tritangent planes.

5. Determination of the remaining 30 tritangent planes.

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* Math. Ann. Vol. 44, 1894, pp. 375-412.

1. DEVELOPMENT OF THE COORDINATE SYSTEM.

Let the three skew lines l_1, l_2, l_3 be the axes of pencils of planes, tangent to the quadric surface determined by l_1, l_2, l_3. We will establish a projective scale on l_2 and l_3 by giving points on the three lines cut out by a common cross generator, the same coordinate. Let the parameters of the points of l_1, be $x_1 : x_2$, those of l_2 be $y_1 : y_2$ and those of l_3 be $z_1 : z_2$. Planes of the pencils will have the parameters of the cross lines. In general fixing values of these three ratios fixes a unique point in space and a generic point in space determines a unique plane in each pencil and hence unique values of the ratios. Two indeterminate cases arise. First in case $x_1 : x_2 = y_1 : y_2 = z_1 : z_2$, the point expands into a line of the cross regulus. Secondly, when we have given a point on one of the fixed lines, the plane of that line is indeterminate. Certain relations exist between the six coordinates x_1, x_2, y_1, y_2, z_1, z_2 for there exist values of λ, μ, ν such that

$$\lambda(xt) + \mu(yt) + \nu(zt) \equiv 0 \qquad \text{for all } t,$$

which by the transformation $(x^1t) = \lambda(xt)$, $(y^1t) = \mu(yt)$, $(z^1t) = \nu(zt)$ becomes $(xt) + (yt) + (zt) \equiv 0$ or $x_0 + y_0 + z_0 = 0$ and $x_1 + y_1 + z_1 = 0$

Any plane in space can be written in the form —

$$(\xi x) + (\eta y) + (\zeta z) = 0 .$$

By adding multiples of the above identity we can add the condition that —

$$(\xi t) + (\eta t) + (\zeta t) \equiv 0 \qquad \text{for all } t.$$

If we connect the planes on the lines l_1, l_2, l_3 by the relation, $(\alpha x)(\beta y)(\gamma z) = 0$, we obtain a cubic surface containing the three fixed lines $(xt) \equiv 0$, $(yt) \equiv 0$, $(zt) \equiv 0$. In case the cubic sur-

face is degenerate it is composed of the quadric on l_1, l_2, l_3 and some plane in space. In fact any plane in space is representable by a form, $(\alpha x)(\beta y)(\gamma z) = 0$, out of which we can factor the fixed quadric. In general a surface of the $_n$th order is represented by the form, $(\alpha x)^n(\beta y)^n(\gamma z)^n = 0$, out of which the fixed quadriccan be factored n times.

2. THE GROUP OF THE LINES.

It will be convenient to use the double six notation for the lines of the surface. Let l_i and m_i, i = 1, 2, 3, 4, 5, 6, , be the two sets of six skew lines in the double six such that m_i does not intersect l_i and let $l_{ij} = l_{ji}$, i = 1, 2 - - - 6, j = 1, 2 - - - -6 but i j, be the line which intersects l_i, l_j, m_i, m_j, and l_{mk}.

The group of the lines of a cubic surface is of order 51, 840*. Since the surface defined by the trilinear form contains the three fixed lines, the group of its lines is reduced. Fixing one line l_1, the group which permutes the remaining lines is of order 1920. There are in this group two transitive sets.

There are sixteen lines which do not intersect l_1, so that the group of the lines when two non-intersecting lines l_1 and l_2 are each fixed is of order 120. Of the sixteen lines which do not intersect l_2, one is l_1 and five intersect l_1, so there are ten lines which intersect neither l_1 nor l_2. So fixing each of three lines l_1, l_2, l_3 the group of permutations of the lines is a G_{12}. This group is the product of a G_6 and a G_2. The three lines m_4, m_5, m_6 all intersect l_1, l_2, l_3. That these are

*Pascal's Repertorium der Höheren Mathematik II p. 287
Jordan, Traité des Substitutions et des Équations Algébriques p. 318.

the only lines intersecting all the three fixed lines is evident analytically for placing $x = y = z = t$ in the equation of the surface we obtain a cubic $(\pi t)^3$. The roots of this cubic give the parameters of the three cross lines. Denote by t_i the parameter of the line m_i, $i = 4, 5, 6$. The group of permutations of these lines form the G_6.

Any plane through l_1, for example $(\lambda x) = 0$, intersects the cubic surface in a conic, namely $(\alpha\lambda)(\beta y)(yz) = 0$, which in general intersects l_1 in two points, and intersects l_2 and l_3 each in one point. We will ask in what cases this conic breaks up into two lines. This will be so when the determinant of $(\alpha\lambda)(\beta y)(yz) = 0$, considered as a bilinear equation in y and z, vanishes. This determinant is of the second degree in λ. That is there are two planes on l_1, which are tritangent planes of the surface, each containing a line intersecting l_1 and l_2 and one intersecting l_1 and l_3. We will speak of these planes as the planes $(\lambda_1 x) = 0$ and $(\lambda_2 x) = 0$. From the symmetry of the form there are two corresponding tritangent planes on l_2, which we denote by $(\mu_1 y) = 0$ and $(\mu_2 y) = 0$ and two on l_3 which will be denoted by $(\nu_1 z) = 0$ and $(\nu_2 z) = 0$ These planes are invariant under the G_6 which permutes the roots of the cubic. There is a group G_2 which interchanges the planes $(\lambda_1 x) = 0$ and $(\lambda_2 x) = 0$. This G_2 changes the sign of the discriminant of the quadratic in λ and also of the quadratics in μ and ν and therefore the discriminants differ only by rational factors. Hence we can conclude that the G_2 also interchanges the planes $(\mu_1 y) = 0$ and $(\mu_2 y) = 0$ and the planes $(\nu_1 z) = 0$ and $(\nu_2 z) = 0$. This is actually verified by showing the identity of the discriminants of the three quadratics. We will expand the trilinear form by the Clebsch-Gordan expansion thus: -

$$(\lambda x)(\mu y)(yz) = (\pi x)(\pi y)(\pi z) + (\rho x)(yz) + (\sigma y)(zx) + (\tau z)(xy).$$

where $(\rho x) + (\sigma x) + (\tau x) \equiv 0$

The identity of the discriminants will be proven if we prove that the discriminant of any one of the three quadratics is symmetrical in ρ, σ, τ. We will take the quadratic in x which is written symbolically :-

$$\tfrac{1}{2}(\pi\pi')^2(\pi x)(\pi' x) + (\rho x)^2 - (\rho x)(\sigma x) - (\rho x)(\tau x) + (\sigma x)(\tau x) + (\pi x)^2[(\pi\sigma) - (\pi\tau)].$$

Its discriminant is -

$$\tfrac{1}{4}(\pi\pi')^2(\pi'\pi'')^2(\pi\pi'')(\pi'\pi''') + (\pi\pi'')^2[(\pi\sigma) - (\pi\tau)][(\pi''\sigma) - (\pi''\tau)] + (\pi\pi')^2(\pi\sigma)(\pi'\tau) + 2(\pi\pi')^2(\pi\rho)(\pi'\rho) - 2(\pi\sigma)(\pi\tau)[(\pi\sigma) - (\pi\tau)] - 4(\pi\rho)^2[(\pi\sigma) - (\pi\tau)] + 4(\sigma\rho)(\tau\rho) - \tfrac{1}{2}(\sigma\tau)^2.$$

To show it symmetrical in ρ, σ, τ we consider first the term of the third degree $\frac{1}{4}(\pi\pi')^2(\pi''\pi''')^2(\pi\pi'')(\pi'\pi''')$. This is symmetrical being the discriminant of the cubic $(\pi x)^3$. The terms of the second degree, $(\pi\pi')^2[(\pi\sigma) - (\pi\tau)][(\pi'\sigma) - (\pi'\tau)] + (\pi\pi')^2(\pi\sigma)(\pi'\tau) + 2(\pi\pi')^2(\pi\rho)(\pi'\rho)$ forms the bilinear invariant of $(\pi\pi')^2(\pi x)(\pi' x)$ with $[(\sigma x) - (\tau x)]^2 + (\sigma x)(\tau x) + 2(\rho x)^2$ which by use of the identity in ρ, σ, τ reduces to $\tfrac{3}{2}[(\sigma x)^2 + (\tau x)^2 + (\rho x)^2]$, an expression symmetrical in ρ, σ, τ. The terms $-2(\pi\sigma)(\pi\tau)[(\pi\sigma) - (\pi\tau)] - 4(\pi\rho)^2[(\pi\sigma) - (\pi\tau)]$ is the bilinear invariant of $(\pi x)^3$ with $-2(\sigma x)(\tau x)[(\sigma x) - (\tau x)] - 4(\rho x)^2[(\sigma x) - (\tau x)]$ which can be put in the form, $2[(\sigma x) - (\tau x)][(\tau x) - (\rho x)][(\rho x) - (\sigma x)]$, which is not altered by permuting ρ, σ, τ. Finally $4(\sigma\rho)(\tau\rho) - 1/2(\sigma\tau)^2 = -3/2[(\sigma\tau)^2 + (\rho\sigma)^2 + (\tau\rho)^2]$ since $(\rho\sigma) = (\sigma\tau) = (\tau\rho)$. The transformation of the lines and planes by the G_{12} can be indicated in the following manner, using the double six notation. Conjugate sets of lines under G_{12} are represented in the braces where the G_6 permutes the

lines horizontally and the G_2 interchanges rows.

$$\begin{pmatrix} l_1 \\ l_1 \end{pmatrix} \quad \begin{pmatrix} l_2 \\ l_2 \end{pmatrix} \quad \begin{pmatrix} l_3 \\ l_3 \end{pmatrix} \cdot \quad \begin{pmatrix} m_4 & m_5 & m_6 \end{pmatrix}$$

$$\begin{Bmatrix} l_{14} & l_{15} & l_{16} \end{Bmatrix} \quad \begin{Bmatrix} l_{24} & l_{25} & l_{26} \end{Bmatrix} \quad \begin{Bmatrix} l_{34} & l_{35} & l_{36} \end{Bmatrix}$$

$$\begin{pmatrix} l_4 & l_5 & l_6 \\ l_{56} & l_{64} & l_{45} \end{pmatrix} \quad \begin{pmatrix} m_1 \\ l_{23} \end{pmatrix} \quad \begin{pmatrix} m_2 \\ l_{31} \end{pmatrix} \quad \begin{pmatrix} m_3 \\ l_{12} \end{pmatrix} .$$

We will notice that the G_2 converts the double six

$$D = \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \end{pmatrix} \quad \text{into}$$

$$D^1 \equiv \begin{pmatrix} l_1 & l_2 & l_3 & l_{56} & l_{64} & l_{45} \\ l_{23} & l_{31} & l_{12} & m_4 & m_5 & m_6 \end{pmatrix} .$$

The conjugate sets of tritangent planes are as follows where $T_{12,34,56}$ contains the lines l_{12}, l_{34}, l_{56} and $T_{1234,5}$ contains the lines l_6, m_5, l_{56}.

$$\begin{pmatrix} T_{14,25,36} & T_{15,26,34} & T_{16,24,35} \end{pmatrix} \quad \begin{pmatrix} T_{14,26,35} & T_{15,24,36} & T_{16,25,34} \end{pmatrix}$$

$$\begin{pmatrix} T_{1245,6} & T_{1256,4} & T_{1264,5} \end{pmatrix} \quad \begin{pmatrix} T_{1345,6} & T_{1356,4} & T_{1364,5} \end{pmatrix}$$

$$\begin{pmatrix} T_{2345,6} & T_{2356,4} & T_{2364,5} \end{pmatrix} \quad \begin{pmatrix} T_{1456,2} \\ T_{2456,1} \end{pmatrix} \quad \begin{pmatrix} T_{1456,3} \\ T_{3456,1} \end{pmatrix} \quad \begin{pmatrix} T_{2456,3} \\ T_{3456,2} \end{pmatrix}$$

$$\begin{pmatrix} T_{12,34,56} & T_{12,35,64} & T_{12,36,45} \\ T_{1256,3} & T_{1264,3} & T_{1245,3} \end{pmatrix} \quad \begin{pmatrix} T_{13,24,56} & T_{13,25,64} & T_{13,26,45} \\ T_{1356,2} & T_{1364,2} & T_{1345,2} \end{pmatrix}$$

$$\begin{pmatrix} T_{23,14,56} & T_{23,15,64} & T_{23,16,45} \\ T_{2356,1} & T_{2364,1} & T_{2345,1} \end{pmatrix} \quad \begin{pmatrix} T_{1234,5} & T_{1235,6} & T_{1236,4} \\ T_{1236,5} & T_{1234,6} & T_{1235,4} \end{pmatrix}$$

3. EQUATIONS OF FIFTEEN TRITANGENT PLANES.

The equations of fifteen tritangent planes to the surface can be obtained immediately from the trilinear form. For the three lines m_4, m_5, m_6, which intersect l_1, l_2, l_3, we have the parameters t_4, t_5, t_6, obtainable in terms of the constants of the equation by the solution of the cubic $(\Pi x)^3$. The equation of nine planes are evident at once for if we let i = 4, 5, 6, the tritangent plane containing l_1 and m_i is $(t_i x) = 0$, that containing l_2 and m_i is $(t_i y) = 0$ and that containing l_3 and m_i is $(t_i z) = 0$. Each of these planes contains a third line of the surface. We will denote by l_{ij} the third line in the plane on l_i and m_i where i = 1, 2, 3, and j = 4, 5, 6. Also we will denote by t_{ij} the parameter of the point in which l_{ij} meets l_i. In terms of these parameters the equations of the remaining six planes take an exceptionally simple form. An explicit form for t_{ij} in terms of the roots t_4, t_5, t_6 and the constants of the equation is readily obtainable. It will be sufficient to obtain this result for one parameter, eg. t_{14}. By solving between the plane $(t_4 x) = 0$ and the trilinear form and setting z = y, we obtain a quadratic in y which has the form $(t_4 y)(t_{14} y) = 0$. To do this we will develop the trilinear form by the Clebsch-Gordan expansion thus -

$$(1)\quad (\alpha x)(\beta y)(\gamma z) = \tfrac{1}{3}\{(t_4 x)(t_5 y)(t_6 z) + (t_5 x)(t_6 y)(t_4 z) + (t_6 x)(t_4 y)(t_5 z)\} + \tfrac{1}{3}\{(\gamma z)(l x) + (z x)(m y) + (x y)(n z)\}$$

in which the identity $(lx) + (my) + (nz) \equiv 0$ holds. Hence

$$(2)\quad (\alpha t_4)(\beta y)(\gamma z) = \tfrac{1}{3}\{(t_5 t_4)(t_6 y)(t_4 z) + (t_6 t_4)(t_4 y)(t_5 z)\} + \tfrac{1}{3}\{(\gamma z)(l t_4) + (z t_4)(m y) + (t_4 y)(n z)\}.$$

Placing z = y

$$(3) \qquad (\alpha t_4)(\beta y)(yz) = 1/3\,(t_4 y)\left\{(t_6 t_4)(t_5 y) - (t_4 t_5)(t_6 y)\right\} - \left\{(my) - (ny)\right\}$$

and therefore –

$$(4) \quad t_{14} = 1/3\left[\left\{(t_6 t_4)t_5 - (t_4 t_5)t_6\right\} - \left\{m-n\right\}\right].$$

If we represent the cubic covariant points of the cubic $(\pi x)^3$ by $3q_4$, $3q_5$, $3q_6$ and the cubic covariant points of the cubic whose roots are l, m, n by $3q_1$, $3q_2$, $3q_3$ then we have –

$$(5) \quad \begin{cases} q_4 = \frac{1}{3}\{(t_6 t_4)t_5 - (t_4 t_5)t_6\} \\ q_5 = \frac{1}{3}\{(t_4 t_5)t_6 - (t_5 t_6)t_4\} \\ q_6 = \frac{1}{3}\{(t_5 t_6)t_4 - (t_6 t_4)t_5\} \end{cases}.$$

$$(6) \quad \begin{cases} q_1 = 1/3\,(m-n) \\ q_2 = 1/3\,(n-l) \\ q_3 = 1/3\,(l-m). \end{cases}$$

So $t_{14} = q_4 - q_1$ and in general –

$$(7) \quad t_{ij} = q_j - q_i \qquad \text{where } j = 4,5,6 \qquad i = 1,2,3,$$

Let us note that –

$$(8) \quad q_4 + q_5 + q_6 = 0$$

$$(9) \quad q_1 + q_2 + q_3 = 0$$

The nine linear forms, $(t_{ij}t)$, satisfy 6 linear relations represented thus: if we consider the determinant,

$$\begin{Bmatrix} (t_{14}t) & (t_{24}t) & (t_{34}t) \\ (t_{15}t) & (t_{25}t) & (t_{35}t) \\ (t_{16}t) & (t_{26}t) & (t_{36}t) \end{Bmatrix},$$

the sum of the three forms which make up any term is zero. It is evident that only five of these relations are independent. Only four of the nine forms are linearly independent. This we can write -

$$(10) \qquad \left\{ \begin{array}{l} (t_{i5}t) - (t_{i6}t) = -(t_5t_6)(t_4t) \\ (t_{i6}t) - (t_{i4}t) = -(t_6t_4)(t_5t) \\ (t_{i4}t) - (t_{i5}t) = -(t_4t_5)(t_6t) \end{array} \right\} \qquad i = 1, 2, 3.$$

and -

$$(11) \qquad \left\{ \begin{array}{l} (t_{2j}t) - (t_{3j}t) = (lt) \\ (t_{3j}t) - (t_{1j}t) = (mt) \\ (t_{1j}t) - (t_{2j}t) = (nt) \end{array} \right\} \qquad j = 4, 5, 6.$$

while -

$$(12) \qquad (t_{ij}t) + (t_{kl}t) + (t_{mn}t) \equiv 0 \qquad \text{when} \quad \begin{array}{l} (i, k, m = 1, 2, 3. \\ (j, l, n = 4, 5, 6. \end{array}$$

The line t_{14} can be determined by two tangent planes to the quadric, one being $(t_4x) = 0$, and the other, containing the generator t_{14}, is of the form -

$$(13) \qquad \rho(t_{14}x) + \sigma(t_{14}y) + \tau(t_{14}z) = 0.$$

Since the meet of this plane with the plane, $(t_4x) = 0$, and the cubic surface is to be a line instead of two points, we can eliminate x and y between the three equations and put the condition on the resulting equation that it be an identity in z. By use of the identity (12), equation (13) becomes -

$$(14)\qquad (t_{14}y) = -\frac{(\sigma + 2\tau)}{(2\sigma + \tau)}(t_{14}z)$$

and $(t_4x) = 0$ may be written –

$$(15)\qquad (t_4y) = -(t_4z).$$

We will eliminate y between (14), (15) and $(\alpha t_4)(yz)(\beta y) = 0$.

Whence –

$$(16)\qquad \frac{\sigma + 2\tau}{2\sigma + \tau}(t_{14}z)(\alpha t_4)(\beta t_4)(yz) - (t_4z)(\alpha t_4)(\beta t_{14})(yz) \equiv 0$$

From (2) we obtain –

$$(17)\qquad (\alpha t_4)(\beta y)(yz) = \tfrac{1}{2}\left[(t_4y)(t_{14}z) + (t_{14}y)(t_{14}z)\right] + \tfrac{1}{2}(lt_4)(yz)$$

and therefore –

$$(18)\qquad (\alpha t_4)(\beta t_4)(yz) = 1/2\,(t_4z)\left\{(t_{14}t_4) + (lt_4)\right\}$$

$$(19)\qquad (\alpha t_4)(\beta t_{14})(yz) = 1/2\,(t_{14}z)\left\{(t_4t_{14}) + (lt_4)\right\}$$

Substituting (18) and (19) in (6),

$$(20)\qquad (t_4z)(t_{14}z)\left\{(3\sigma + 3\tau)(t_{14}t_4) - (\sigma - \tau)(lt_4)\right\} \equiv 0.$$

That is,

$$(3\sigma + 3\tau)(t_{14}t_4) - (\sigma - \tau)(lt_4) = 0$$

Therefore –

$$\rho = 2(lt_4), \qquad \sigma = 3(t_4t_{14}) - (lt_4), \qquad \tau = -3(t_4t_{14}) - (lt_4).$$

Hence the equation of the tangent plane, (13), is –

$$(21)\qquad 2(lt_4)(t_{14}x) + \left[3(t_4t_{14}) - (lt_4)\right](t_{14}y) - \left[3(t_4t_{14}) + (lt_4)\right](t_{14}z) = 0$$

In general the equation of the plane on the line l_{ij} and the generator t_{ij} is obtained from (21) by permuting x, y, z, the symbols l, m, n, and the subscripts 1, 2, 3, as i = 1, 2, 3, and by permuting the subscripts 4, 5, 6, as j = 4, 5, 6.

The six planes to determine are those on

$$l_{14}, l_{25}, l_{36}$$
$$l_{24}, l_{35}, l_{16}$$
$$l_{34}, l_{15}, l_{26}$$
$$l_{14}, l_{26}, l_{35}$$
$$l_{24}, l_{15}, l_{36}$$
$$l_{34}, l_{25}, l_{16} .$$

The plane on l_{14}, l_{25}, l_{36} since it belongs to the pencil on l_{14} and to the pencil on l_{25} is of the form -

$$\lambda_1(t_4x) + \mu_1\Big[2(lt_4)(t_{14}x) + \Big\{3(t_4t_{14})-(lt_4)\Big\}(t_{14}y)-\Big\{3(t_4t_{14})+(lt_4)\Big\}(t_{14}z)\Big] = 0$$

and also has the form -

$$\lambda_2(t_5y) + \mu_2\Big[\Big\{3(t_5t_{25})-(mt_5)\Big\}(t_{25}x)+ 2(mt_5)(t_{25}y)+ \Big\{3(t_5t_{25})-(mt_5)\Big\}(t_{25}z) = 0.$$

Equating coefficients in the two forms we find -

$$\lambda_1 = 2\,(lt_4)(t_{14}t_{25}), \qquad \mu_1 = -\,(t_4t_{25}).$$

By putting in these values and making use of the identity (2) we find the tritangent plane l_{14}, l_{25}, l_{36} takes any one of the three forms.

$$(t_{25}x) - (t_{14}y) = 0$$
$$(t_{36}x) - (t_{14}z) = 0$$
$$(t_{36}y) - (t_{25}z) = 0 .$$

In general the equation of the plane on l_{1j}, l_{2l}, l_{3n} where j, l, n = 4, 5, 6, may be written –

$$(t_{2l}x) - (t_{1j}y) = 0$$

$$(t_{3n}x) - (t_{1j}z) = 0$$

$$(t_{3m}y) - (t_{2l}z) = 0 \ .$$

In terms of the cubic covariant points we have –

$l_{14}\ l_{25}\ l_{36}$ $\quad (q_6y) - (q_3y) - (q_5z) + (q_2z) = 0$

$l_{14}\ l_{26}\ l_{35}$ $\quad (q_6z) - (q_2z) - (q_5y) + (q_3y) = 0$

$l_{15}\ l_{26}\ l_{34}$ $\quad (q_4y) - (q_3y) - (q_6z) + (q_2z) = 0$

$l_{15}\ l_{24}\ l_{36}$ $\quad (q_4z) - (q_2z) - (q_6y) + (q_3y) = 0$

$l_{16}\ l_{24}\ l_{35}$ $\quad (q_5y) - (q_3y) - (q_4z) + (q_2z) = 0$

$l_{16}\ l_{25}\ l_{34}$ $\quad (q_5z) - (q_2z) - (q_4y) + (q_3y) = 0 \ .$

If the surface be expressed in the Cremona hexahedral form, $a^3 + b^3 + c^3 + d^3 + e^3 + f^3$, where a, b, - - - f are six planes connected by the identities, $a + b + c + d + e + f \equiv 0$, $\bar{a}a + \bar{b}b + \bar{c}c + \bar{d}d + \bar{e}e + \bar{f}f \equiv 0$ and $\bar{a} + \bar{b} + \bar{c} + \bar{d} + \bar{e} + \bar{f} = 0$, expressions for the forty-five tritangent planes have been given by Professor Coble*. These forms will be useful in finding the equations of the remaining thirty planes. The fifteen planes found thus far can be exhibited in terms of the six planes, a, b - - - - f, –

$a + d = (t_5t_6)(t_4x)$, $\quad b + e = (t_5t_6)(t_4y)$, $\quad c + f = (t_5t_6)(t_4z)$

$b + f = (t_6t_4)(t_5x)$, $\quad c + d = (t_6t_4)(t_5y)$, $\quad a + e = (t_6t_4)(t_5z)$

$c + e = (t_4t_5)(t_6x)$, $\quad a + f = (t_4t_5)(t_6y)$, $\quad b + d = (t_4t_5)t_6z)$,

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* Trans. Amer. Math. Soc. Vol. XVI. April 1915, p. 173

where $a + b + c + d + e + f \equiv 0$,

$e + f = (t_{36}y)-(t_{25}z) = (t_{14}z) -(t_{36}x) = (t_{25}x) -(t_{14}y)$, $\overline{l_{14}\ l_{25}\ l_{36}}$

$b + c = (t_{26}z)-(t_{35}y) = (t_{35}x) -(t_{14}z) = (t_{14}y) -(t_{26}z)$, $\overline{l_{14}\ l_{26}\ l_{35}}$

$d + e = (t_{34}y)-(t_{26}z) = (t_{15}z) -(t_{34}x) = (t_{26}x) - (t_{15}y)$, $\overline{l_{15}\ l_{26}\ l_{34}}$

$a + c = (t_{24}z)-(t_{36}y) = (t_{36}x) -(t_{15}z) = (t_{15}y) -(t_{24}x)$, $\overline{l_{15}\ l_{24}\ l_{36}}$

$d + f = (t_{35}y)-(t_{24}z) = (t_{16}z) -(t_{35}x) = (t_{24}x) -(t_{16}y)$, $\overline{l_{16}\ l_{24}\ l_{35}}$

$a + b = (t_{25}z)-(t_{34}y) = (t_{34}x) -(t_{16}z) = (t_{16}y) -(t_{25}x)$. $\overline{l_{16}\ l_{25}\ l_{34}}$

Solving these we find -

$2a = (t_{15}y) + (t_{25}y) - (t_{16}z) - (t_{36}z) = 2(q_5y) + (q_3y) - 2(q_6z) - (q_2z)$,

$2b = (t_{25}z) + (t_{35}z) - (t_{26}x) - (t_{16}x) = 2(q_6y) + (q_3y) - 2(q_4z) -(q_2z)$,

$2c = (t_{35}x) + (t_{15}x) - (t_{36}y) - (t_{26}y) = 2(q_4y) + (q_3y) - 2(q_5z) - (q_2z)$,

$2d = (t_{15}z) + (t_{35}z) - (t_{16}y) - (t_{26}y) = -2(q_6y) - (q_3y) + 2(q_5z) + (q_2z)$,

$2e = (t_{25}x) + (t_{15}x) - (t_{26}z) - (t_{36}z) = -2(q_5y) - (q_3y) + 2(q_4z) + (q_2z)$,

$2f = (t_{35}y) + (t_{25}y) - (t_{36}x) - (t_{16}x) = -2(q_4y) - (q_3y) + 2(q_6z) + (q_2z)$.

The condition that -

$$\bar{a}a + \bar{b}b + \bar{c}c + \bar{d}d + \bar{e}e + \bar{f}f \equiv 0$$

is equivalent to the conditions -

$2(\bar{a} - \bar{c} - \bar{e} + \bar{f})\, q_5 + 2(\bar{b} - \bar{c} - \bar{d} + \bar{f})\, q_6 + 2(\bar{a} + \bar{b} + \bar{c})\, q_3 = 0$

$2(\bar{c} - \bar{b} - \bar{d} + \bar{e})\, q_5 + 2(\bar{a} - \bar{b} + \bar{e} - \bar{f})\, q_6 + 2(\bar{a} + \bar{b} + \bar{c})\, q_2 = 0$

and hence we have the six equations -

$$\bar{a} + \bar{b} + \bar{c} = (q_5q_6) = (q_4\, q_5) = (q_6q_4)$$

$$\bar{a} - \bar{c} - \bar{e} + \bar{f} = (q_6q_3)$$

$$\bar{c} + \bar{d} - \bar{b} - \bar{f} = (q_5q_3)$$

$$\bar{c} - \bar{b} - \bar{d} + \bar{e} = (q_6q_2)$$

$$\bar{b} + \bar{f} - \bar{a} - \bar{e} = (q_5q_2)$$

$$\bar{a} + \bar{b} + \bar{c} + \text{---} + \bar{f} = 0$$

Solving we obtain the result -

$$3\bar{a} = (q_5q_6) + (q_6q_3) - (q_5q_2)$$

$$3\bar{b} = (q_5q_6) + (q_4q_3) - (q_6q_2)$$

$$3\bar{c} = (q_5q_6) + (q_5q_3) - (q_4\ q_2)$$

$$3\bar{d} = -(q_5q_6) - (q_6q_3) + (q_5q_1)$$

$$3\bar{e} = -(q_5q_6) - (q_5q_3) + (q_4q_1)$$

$$3\bar{f} = -(q_5q_6) - (q_4q_3) + (q_6q_1).$$

According to the paper by Professor Coble, previously cited, the remaining thirty planes take three equivalent forms -

$$(\bar{be} - d_2)(b + e) - (\bar{cf} + d_2)(c + f) = 0$$

$$(\bar{cf} - d_2)(c + f) - (\bar{ad} + d_2)(a + d) = 0$$

$$(\bar{ad} - d_2)(a + d) - (\bar{be} + d_2)(b + e) = 0$$

in which -

$$\bar{cf} = a_2 + 2(\bar{c}^2 + \bar{f}^2 + \bar{cf}) \text{ and}$$

$$a_2 = \sum \bar{ab}$$

$$-d_2^2 = \bar{be}\ \bar{cf} + \bar{cf}\ \bar{ad} + \bar{ad}\ \bar{be}$$

With reference to the sets of nine and six planes, these thirty separate into three sets. There are 18 planes of the form -

$$\bar{ab}(a + b) + \bar{df}(d + f) - d_2\{(a + b) + (d + f)\} = 0,$$

6 planes of the form -

$$\bar{ad}(a + d) - \bar{bf}(b + f) - d_2\{(a + d) + (b + f)\} = 0,$$

and 6 planes of the form -

$$\bar{ad}(a + b) - \bar{be}(b + e) - d_2\{(a + d) + (b + e)\} = 0.$$

4. A NEW CANONICAL FORM FOR THE CUBIC SURFACE.

The remaining thirty planes are expressible in terms of q_1, q_2, q_3, q_4, q_5, q_6 by mere substitution. However, due to the identities involved, the same plane can have various different forms, all equivalent and no one of more importance than the others. In order to obtain a single equation for each plane we will introduce new variables $\bar{x}$ and $\bar{y}$ such that –

$$(xt) + (yt) + (zt) = 0$$
$$(xt) + \omega(yt) + \omega^2(zt) = 3(\bar{x}t)$$
$$(xt) + \omega^2(yt) + \omega(zt) = 3(\bar{y}t) .$$

Inversely –

$$(xt) = (\bar{x}t) + (\bar{y}t)$$
$$(yt) = \omega^2(\bar{x}t) + \omega(\bar{y}t)$$
$$(zt) = \omega(\bar{x}t) + \omega^2(\bar{y}t) .$$

If we consider the cubic $(\pi t)^3$, polarized with respect to x, y, z and make the above transformation then we have –

$$(\pi t)^3_{x,y,z} = (\pi x)(\pi y)(\pi z) = (\pi \bar{x})^3 + (\pi \bar{y})^3 .$$

But if –

$$(t_5t_6)(t_4y) = (\omega-\omega^2)\left[(h_1y) + (h_2y)\right] ,$$
$$(t_6t_4)(t_5y) = (\omega-\omega^2)\left[\omega^2(h_1y) + \omega(h_2y)\right] ,$$
$$(t_4t_5)(t_6y) = (\omega-\omega^2)\left[\omega(h_1y) + \omega^2(h_2y)\right] .$$

where h_1 and h_2 are the Hessian points of $(\pi t)^3$, then –

$$(\pi t)^3_{x,y,z} = \left[(t_4t)(t_5t)(t_6t)\right]_{x,y,z} = \frac{3(\omega^2-\omega)}{(t_4t_5)(t_5t_6)(t_6t_4)} \left[(h_1t)^3 + (h_2t)^3\right]_{x,y,z}$$

and therefore –

$$(\pi t)^3_{x,y,z} = \frac{3(\omega^2-\omega)}{(t_4t_5)(t_5t_6)(t_6t_4)} \left[(h_1\bar{x})^3 + (h_2\bar{x})^3 + (h_1\bar{y})^3 + (h_2\bar{y})^3\right]$$

By the transformation, –

$$(yz)(lx) = (\omega-\omega^2)(\bar{x}\bar{y})\left[(l\bar{x}) + (l\bar{y})\right],$$

$$(zx)(my) = (\omega-\omega^2)(\bar{x}\bar{y})\left[\omega^2(m\bar{x}) + \omega(m\bar{y})\right],$$

$$(xy)(nz) = (\omega-\omega^2)(\bar{x}\bar{y})\left[\omega(n\bar{x}) + \omega^2(n\bar{y})\right].$$

Similarly for the cubic roots, l, m, n, we write –

$$-l = q_2 - q_3 = k_1 + k_2,$$

$$-m = q_3 - q_1 = \omega k_1 + \omega^2 k_2,$$

$$-n = q_1 - q_2 = \omega^2 k_1 + \omega k_2,$$

in which k_1 and k_2 are the Hessian points of the cubic having roots l, m, n.

Thus –

$$q_1 = -\frac{1}{\omega^2-\omega}(k_1 - k_2)$$

$$q_2 = -\frac{1}{\omega^2-\omega}(\omega k_1 - \omega^2 k_2)$$

$$q_3 = -\frac{1}{\omega^2-\omega}(\omega^2 k_1 - \omega k_2).$$

Taking the expanded trilinear form, –

$$(\alpha x)(\beta y)(\gamma z) = \frac{3(\omega^2-\omega)}{(t_4t_5)(t_5t_6)(t_6t_4)}\left[(h_1\bar{x})^3 + (h_2\bar{x})^3 + (h_1\bar{y})^3 + (h_2\bar{y})^3\right]$$
$$+(\omega^2-\omega)(\bar{x}\bar{y})\left[(k_1\bar{x}) + (k_2\bar{y})\right].$$

If we introduce new variables by setting –

$$(h_1\bar{x}) = X_0$$

$$(h_2\bar{x}) = X_1$$

$$(h_1\bar{y}) = Y_0$$

$$(h_2\bar{y}) = Y_1$$

and use the fundamental identity, then –

$$(h_1h_2)(\bar{x}\bar{y}) = (h_1\bar{x})(h_2\bar{y}) - (h_2\bar{x})(h_1\bar{y}) = (XY),$$

$$(h_1h_2)(k_1\bar{x}) = (h_1k_1)(h_2\bar{x}) - (h_2k_1)(h_1\bar{x}) = (h_1k_1)\,X_1 - (h_2k_1)\,X_0,$$

$$(h_1h_2)(k_2\bar{y}) = (h_1k_2)(h_2\bar{y}) - (h_2k_2)(h_1\bar{y}) = (h_1k_2)\,Y_1 - (h_2k_2)\,Y_0.$$

Since

$$q_4 = 1/3\,\{(t_6t_4)t_5 - (t_4t_5)t_6\}$$

$$q_5 = 1/3\,\{(t_4t_5)t_6 - (t_5t_6)t_4\}$$

$$q_6 = 1/3\,\{(t_5t_6)t_4 - (t_6t_4)t_5\},$$

then

$$q_4 = (h_1 - h_2)$$

$$q_5 = (\omega^2h_1 - \omega h_2)$$

$$q_6 = (\omega h_1 - \omega^2h_2)$$

and by taking the bilinear invariant,

$$(h_1h_2) = -1/3(\omega^2-\omega)(q_4q_5) = -\frac{1}{3(\omega-\omega^2)}\,(t_4t_5)(t_6t_4)(t_5t_6).$$

If these be substituted in the trilinear form, the result is -

$$(h_1h_2)^2(\alpha x)(\beta y)(\gamma z) = (h_1h_2)\{X_0^3 + Y_0^3 + X_1^3 + Y_1^3\} + (\omega^2-\omega)\,(XY)\,[h_1k_1)X_1 - (h_2k_1)X_0 + (h_1k_2)\,Y_1 - (h_2k_2)Y_0]$$

Writing the constants -

$$(h_1k_1) = a_0 \qquad (h_1k_2) = b_0 \qquad (h_1h_2) = c$$

$$(h_2k_1) = a_1 \qquad (h_2k_2) = b_1$$

the form becomes -

$$c^2(\alpha x)(\beta y)(\gamma z) = c(X_0^3 + Y_0^3 + X_1^3 + Y_1^3) + (\omega^2-\omega)(XY)\,(a_0X_1 - a_1X_0 + b_0Y_1 - b_1Y_0).$$

Obviously this form is not available in case $(h_1h_2) = 0$.

5. THE REMAINING THIRTY TRITANGENT PLANES.

The plane, $a + d = (t_5 t_6)(t_4 x)$, in the new coordinate system becomes

$$(\omega - \omega^2)(X_0 + X_1 + Y_0 + Y_1) = 0.$$

In general the nine planes of the form $a + d$ may be written -

(I) $(\omega - \omega^2)(\omega^k X_0 + \omega^l Y_0 + \omega^{-l} X_1 + \omega^{-k} Y_1) = 0$ where $k = 0, 2, 1$ for the permutations of a, b, c, and $l = 0, 2, 1$ for the permutations of d, f, e. By substituting, in the equations of the remaining six planes already determined, the values of the q's and the new variables, we obtain the following equations; for the three planes of the form $a + b$ we have -

(II) $(\omega^2 - \omega)(\omega^k X_0 + \omega^{-k} Y_1) + \frac{1}{c(\omega^2 - \omega)}(a_0 X_1 - a_1 X_0 + b_0 Y_1 - b_1 Y_0) = 0.$

where $k = 0, 2, 1$ according as a, b, c is omitted.

For the three planes of the form $d + f$ we find -

(III) $(\omega^2 - \omega)(\omega^{-l} X_1 + \omega^l Y_0) - \frac{1}{c(\omega^2 - \omega)}(a_0 X_1 - a_1 X_0 + b_0 Y_1 - b_1 Y_0) = 0,$

where $l = 0, 2, 1$ according as d, f, e is omitted.

In order to write the equations of the remaining thirty planes it will be necessary to calculate a_2 and $\bar{a}$, $\bar{b}$, - - - $\bar{f}$ from their expressions in terms of the q's. The result of such calcuation is -

$$\left.\begin{matrix} 3\bar{a} \\ 3\bar{b} \\ 3\bar{c} \end{matrix}\right) \equiv (\omega^2 - \omega)\left[c + \frac{(\omega^2 - \omega)}{3}\left\{\omega^{-i+1} a_1 - \omega^{i+2} b_0\right\}\right],$$

where $i = 1, 0, 2$ for $\bar{a}$, $\bar{b}$, $\bar{c}$,

$$\left.\begin{matrix} 3\bar{d} \\ 3\bar{f} \\ 3\bar{e} \end{matrix}\right) = (\omega^2 - \omega)\left[-c + \frac{(\omega^2 - \omega)}{3}\left\{\omega^{-j} b_1 - \omega^j a_0\right\}\right],$$

where $j = 1, 0, 2$ for $\bar{d}$, $\bar{f}$, $\bar{e}$,

$$a_2 = c^2 - 1/6\,(a_0 b_1 - a_1 b_0).$$

The squares are -

$$\left.\begin{matrix} 9\,\bar{a}^2 \\ 9\,\bar{b}^{-2} \\ 9\,\bar{c}^{-2} \end{matrix}\right) = -3c^2 - 2(\omega^2-\omega)\,c\left\{\omega^{-i+1}a_1 - \omega^{i+2}b_0\right\} + \left\{\omega^{i+2}a_1^2 - 2a_1b_0 + \omega^{-i+1}b_0^2\right\},$$

where $i = 1, 0, 2$ for $\bar{a}, \bar{b}, \bar{c}$,

$$\left.\begin{matrix} 9\,\bar{d}^2 \\ 9\,\bar{f}^2 \\ 9\,\bar{e}^{-2} \end{matrix}\right) = -3c^2 + 2(\omega^2-\omega)c\left\{\omega^{-j}b_1 - \omega^{j}a_0\right\} + \left\{\omega^{j}b_1^2 - 2a_0b_1 + \omega^{-j}a_0^2\right\},$$

where $j = 1, 0, 2$ for $\bar{d}, \bar{f}, \bar{e}$.

The products of the form $9\,\bar{a}\bar{b}$ can be written -

$$-3\left[c^2 + \frac{(\omega^2-\omega)}{3}\,c\left\{a_1(\omega^{-i+1} + \omega^{-j+1}) - b_0(\omega^{i+2} + \omega^{j+2})\right\} - 1/3\left\{\omega^{-i-j+2}a_1^2 - (\omega^{-i+j} + \omega^{i-j})\,a_1b_0 + \omega^{i+j+1}b_0^2\right\}\right].$$

The products of the form $9\,\bar{d}\bar{f}$ have the form -

$$-3\left[c^2 - \frac{(\omega^2-\omega)}{3}\,c\left\{b_1(\omega^{-i} + \omega^{-j}) - a_0(\omega^{i} + \omega^{j})\right\} - \frac{1}{3}\left\{\omega^{-i-j}b_1^2 - a_0b_1(\omega^{i-j} + \omega^{-i+j}) + \omega^{i+j}a_0^2\right\}\right].$$

Products of the type $9\,\bar{a}\bar{d}$ have the form -

$$1/3\,c^2 - \frac{2}{9}(\omega^2-\omega)c\left\{\omega^{j}a_0 + \omega^{-i+1}a_1 - \omega^{i+2}b_0 - \omega^{-j}b_1\right\} - \frac{11}{18}\left\{a_0b_1 + a_1b_0\right\} + \frac{2}{9}\left\{\omega^{i+2}a_1^2 + \omega^{-i+1}b_0^2 + \omega^{j}b_1^2 + \omega^{-j}a_0^2 + \omega^{-i-j+1}a_1b_1 - \omega^{i-j+2}b_0b_1 - \omega^{-i+j+1}a_0a_1 + \omega^{i+j+2}a_0b_0\right\}.$$

Substituting these results in the expression -

$$\bar{c}\bar{f} = a_2 + 2\left[\bar{c}^{-2} + \bar{f}^{-2} + \bar{c}\bar{f}\right)$$ we find -

any expression of the type, $\bar{a}\bar{b}$, is -

$-c^2 - 1/6\ a_0b_1 - 5/6\ a_1b_0 + 2/3c(\omega^{-k}a_1 - \omega^{k}b_0)$, in which k = 0, 2, 1 according as $\bar{a}$, $\bar{b}$, $\bar{c}$ is omitted.

Any expression of the type, $\overline{df}$, is -

$-c^2 - 5/6\ a_0b_1 - 1/6\ a_1b_0 - 2/3c\ (\omega^{l}b_1 - \omega^{-l}a_0)$, where l = 2, 0, 1 according as $\bar{d}$, $\bar{f}$, $\bar{e}$ is omitted.

Any expression of the type, $\overline{ad}$, is -

$$1/3c^2 - 2/9(\omega^2-\omega)c\left\{\omega^{-l}a_0 + \omega^{-k}a_1 - \omega^{k}b_0 - \omega^{l}b_1\right\} - \frac{11}{18}\left\{a_0b_1+a_1b_0\right\} + 2/9(\omega^{k}a_1^2 + \omega^{-k}b_0^2 + \omega^{-l}b_1^2 + \omega^{l}a_0^2 + \omega^{-k+l}a_1b_1 - \omega^{k+l}b_0b_1 - \omega^{-k-l}a_0a_1 + \omega^{k-l}a_0b_0),$$

where k = 0, 2, 1, as a, b, c are permuted and l = 2, 0, 1, as d, f, e, are permuted.

From these we can write the equations of the eighteen planes of the type -

$$\overline{ab}(a + b) + \overline{df}\ (d + f) - d_2\left[(a+b) + (d+f)\right] = 0,$$

thus -

$$\text{(IV)}\quad 2/3\ \frac{1}{(\omega^2-\omega)}\left[a_0X_1 - a_1X_0 + b_0Y_1 - b_1Y_0\right]$$

$$\left\{\omega^{-k}a_1 - \omega^{k}b_0 + \omega^{l}b_1 - \omega^{-l}a_0 + \frac{1}{c}(a_0b_1 - a_1b_0)\right\} + 2/3(\omega^2-\omega)\left[(\omega^{k}X_0 + \omega^{-k}Y_1)\right.$$

$$\left\{c(\omega^{-k}a_1 - \omega^{k}b_0) - a_1b_0\right\} - (\omega^{l+1}X_1 + \omega^{-l-1}Y_0)\left\{c(\omega^{l}b_1 - \omega^{-l}a_0) + a_0b_1\right\} - (\omega^2-\omega)$$

$$(\omega^{k}X_0 + \omega^{-k}Y_1 + \omega^{l+1}X_1 + \omega^{-l-1}Y_0)\left[c^2 + 1/6\ (a_0b_1 + a_1b_0) + d_2\right] = 0,$$

where k = 0, 2, 1 according as a, b, c is omitted, and l = 2, 0, 1 according as d, f, e, is omitted. The discriminant, d_2, takes both the negative and positive sign.

The six planes of the type -

$$\text{(V)}\quad \overline{ad}(a + d) - \overline{bf}(b + f) - d_2\left[(a + d) + (b + f)\right] = 0\ ,$$

are obtained in the form -

$$\left[\omega X_0 - \omega^{l} X_1 + \omega^{-l} Y_0 - \omega^{2} Y_1\right]\left[c^2 - \frac{11}{6}(a_0 b_1 + a_1 b_0) + 2/3\,\omega^2(\omega^{l+1} b_0 b_1 + \omega^{-l+1} a_0 a_1)\right]$$

$$-2/3\,\omega^2\Big[(X_0 + \omega^{-l+2} Y_0)(a_1^2 + \omega^{-l} b_1^2 - \omega^{l} a_1 b_1) - (X_1 + \omega^{-l+2} Y_1)(\omega^{l} b_0^2 + \omega^{-l} a_0^2 + a_0 b_0)\Big)$$

$$+ d_2(\omega - \omega^2)\left[\omega X_0 + \omega^{l} X_1 + \omega^{-l} Y_0 + \omega^{2} Y_1\right) = 0,$$

where l= 2, 0, 1 as d runs over d, f, e and f runs over f, e, d.

The irrationality, d_2, having a double sign gives two sets.

The six planes of the type -

$$\overline{ad}(a+d) - \overline{be}(b+e) - d_2\left[(a+d)+(b+e)\right] = 0$$

are written thus -

$$\text{(VI)}\quad (\omega X_0 + \omega^{l+2} X_1 - \omega^{-l+1} Y_0 - \omega^{2} Y_1)\left[c^2 - \frac{11}{6}(a_0 b_1 + a_1 b_0) + 2/3\,\omega^2(\omega^{l+1} a_1 b_1 + \omega^{-l+1} a_0 b_0)\right)$$

$$+2/3c(1-\omega)\Big[(b_0 + \omega^{l} b_1)(X_0 + \omega^{l+1} X_1) + (\omega^{l+1} a_0 + \omega^{-l+1} a_1)(Y_0 + \omega^{l+1} Y_1)\Big)$$

$$-2/3\,\omega^2\Big(X_0 + \omega^{l+1} X_1)(a_1^2 + \omega^{l} a_0^2 - \omega^{-l} a_0 a_1) - (Y_0 + \omega^{l+1} Y_1)(\omega^{-l+1} b_0^2 + \omega^{l+1} b_1^2 -$$

$$\omega\, b_0 b_1)\Big] + d_2(\omega - \omega^2)\left[\omega X_0 + \omega^{l+2} X_1 + \omega^{-l+1} Y_0 + \omega^{2} Y_1\right) \neq 0,$$

where l = 2, 0, 1 as d runs over d, f, e and e runs over e, d, f. The discriminant, d_2, takes both the positive and negative sign.

In terms of the three planes -

$$I = a_0 X_1 - a_1 X_0 + b_0 Y_1 - b_1 Y_0$$

$$K = \omega^{k} X_0 + \omega^{-k} Y_1$$

$$L = \omega^{l+1} X_1 + \omega^{-l-1} Y_0 \quad ,$$

the nine planes (I) are -

$$(\omega - \omega^2)(K + L) = 0,$$

the 3 planes (II) are -

$$(\omega^2-\omega)\,K + \frac{1}{c(\omega^2-\omega)}\cdot I = 0 \ ,$$

the 3 planes (III) are -

$$(\omega^2-\omega)\,L - \frac{1}{c(\omega^2-\omega)}\cdot I = 0,$$

and the 18 planes (IV) are -

$$2/3\,\frac{I}{(\omega^2-\omega)}\left\{\omega^{-k}a_1-\omega^{k}b_0+\omega^{1}b_1-\omega^{-1}a_0+\frac{1}{c}(a_0b_1-a_1b_0)\right\}+2/3(\omega^2-\omega)$$

$$\left\{K\left\{c(\omega^{-k}a_1-\omega^{k}b_0)-a_1b_0\right\}-L\left\{c(\omega^{1}b_1-\omega^{-1}a_0)+a_0b_1\right\}\right\}-(\omega^2-\omega)(K+L)$$

$$\left[c^2+1/6(a_0b_1+a_1b_0)+d_2\right]=0.$$

In terms of the two planes -

$$L_0 = X_0 + \omega^{-1+2}Y_0$$

$$L_1 = \omega^{-1}X_1 + \omega^{1+2}Y_1 \ ,$$

the planes (V) are -

$$(\omega L_0-\omega^{-1}L_1)\left[c^2-\frac{11}{6}(a_0b_1+a_1b_0)+2/3(\omega^{1}b_0b_1+\omega^{-1}a_0a_1)\right]-2/3\,L_0\left[c(1-\omega)\right.$$

$$(\omega^{-1}a_0-b_0)+\omega^2(a_1^2+\omega^{-1}b_1^2-\omega^{1}a_1b_1))+2/3L_1\left[c(1-\omega)(\omega^{-1}a_1-b_1)+\omega^2(a_0^2+\omega^{-1}b_0^2+\right.$$

$$\left.\omega^{1}a_0b_0)\right]+d_2(\omega-\omega^2)(\omega L_0+\omega^{-1}L_1)=0.$$

In terms of the two planes -

$$L_0^1 = X_0 + \omega^{1+1}X_1$$

$$L_0^1 = \omega^{1+1}Y_0 + \omega^{-1+2}Y_1,$$

the planes (VI) are -

$$(\omega L_0^1 - \omega^{-1} L_1^1)\left[c^2 - \frac{11}{6}(a_0 b_1 + a_1 b_0) + 2/3(\omega^1 a_1 b_1 + \omega^{-1} a_0 b_0)\right]$$

$$+2/3 L_0^1 (c(1-\omega)(b_0 + \omega^1 b_1) - \omega^2 (a_1^2 + \omega^1 a_0^2 - \omega^{-1} a_0 a_1))$$

$$+2/3 L_1^1 \left[c(1-\omega)(a_0 + \omega^1 a_1) + \omega^2 (b_1^2 + \omega^1 b_0^2 - \omega^{-1} b_0 b_1)\right] + d_2(\omega - \omega^2)\left[\omega L_0^1 + \omega^{-1} L_1^1)\right] = 0.$$

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